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Analytic Continuation By Means Of The L(r,t)Summability Transform

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ANALYTIC CONTINUATION BY MEANS OF THE L(r,t) – SUMMABILITY TRANSFORM

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1. Let $A = (a_{nm})$ and $x = [S_m]$ $(n, m = 0,1,2,...)$ be a matrix and a sequence of complex numbers, respectively. We write

(1) $t_n \equiv A_n(x) = \sum_{m=o}^{\infty} a_{nm} S_m$, and say that the sequence x (and the corresponding series $\sum_{m=o}^{\infty} (S_m - S_{m-1})$, with $S_{-1} = 0$ is summable A to the sum t if each of the series in (1) converges and $\lim_n t_n$ exists and equals t. We say that the method A is regular provided it sums every convergent sequence to its limit. The method A is regular if and only if

(2) $\sum_{m=o}^{\infty} |a_{nm}| \leqslant k \ (n = 0,1,2,...)$,

(3) $\lim_{n\to\infty} a_{nm} = 0 \ (m = 0,1,...)$,

(4) $\lim_{n\to\infty} \sum_{m=o}^{\infty} a_{nm} = 1$

where k is a constant independent of n. These are so called Silverman Toeplitz conditions.

2. The Method L(r,t). For complex r and t denote by L(r,t) the matrix whose elements are $A_{nk} = 0$ when $k < n$, and $A_{nk} = (1 - r)^{n+1} \exp(tr/1 - r) L^n_{k-n} (t) r^{k-n}$, $k \geqslant n$. $L^n_j(t)$ being the Laguerre polynomial of degree j, given by

$$L^n_j (t) = \sum_{i=o}^{j} \binom{j+n}{j-i} \frac{(-t)^i}{i!}$$

Theorem 1. For a given value of z, L(z,t) is regular if and only if $\mathrm{Im}(z) = 0$ and $0 \leqslant \mathrm{Re}(z) < 1$. (R.E. Powell [1])

Theorem 2. Let $|r| < 1$. For each t, the L(r,t) transform continues the geometric series analytically into the region

$$\{z: \left|\frac{(1-r)z}{1-rz}\right| < 1\} \cap \{z: |rz| < 1\}$$

(R.E. Powell, Canad. J. Math. 18(1966), 1251-1260).

3. We shall now investigate summability of Power Series by the L(r,t) method. Let $\sum_{n=o}^{\infty} a_n z^n$ be a power series with finite radius of

convergence R, and let F(z) denote the function obtained by analytic continuation along radial lines from the origin. The region of analyticity of F(z) is the Mittag – Leffler Star S associated with F(z). For each index r and vertex ξ of S, let $C(r,\xi)$ denote the set of points

(5) $|rz| < |\xi|$, $|z - rz| < |\xi - rz|$.

It should be noted that $C(r,\xi)$ is a neighborhood of the origin bounded by circular arcs and having ξ on its boundary.

Now let C(r) denote the set of inner points of the intersection of the sets $C(r,\xi)$ determined by ξ of S. $C(r) \subset S$.

Theorem 3. For each fixed r satisfying $0 < r < 1$, the series $\sum_{n=0}^{\infty} a_n z^n$ is summable L(r,t) to F(z) at every interior point z of the region C(r).

Proof: If z = 0, the theorem is trivially true. Suppose, then, the $z \neq 0$ and z is a fixed point of C(r). Let C_1 denote the circle consisting of points U for which

(6) $|rz| = |U|$ holds, and let C_2 denote the circle consisting of points U for which

(7) $|z - rz| = |U - rz|$ holds.

Inequalities (5) tell us that both C_1 and C_2 are wholly contained inside S. Therefore, a simple closed curve γ can be constructed inside S which contains C_1 and C_2 in its interior.

Since C_1 includes the origin in its interior and z lies on C_2, then γ includės the origin and the point z in its interior. Inside and on γ, F(z) is analytic. Consequently, for each point w on γ the following inequalities hold:

(8) $|F(w)| \leqslant M$, $M = \text{Max } |F(w)|$ for w on γ;

(9) $|w - z| \geqslant N > 0$, $N = \min |w - z|$ for w on γ;

(10) $|rz| < |w|$;

(11)
$$\left|\frac{z - rz}{w - rz}\right| \leq \theta < 1, \quad \theta = \max \left|\frac{z - rz}{w - rz}\right| \quad \text{for } w \text{ on } \gamma.$$

Returning now to the series $\sum_{n=0}^{\infty} a_n z^n$, the coefficients a_n are given by the Cauchy formula

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{F(w)\, dw}{w^{n+1}} \qquad n = 0,1,2,\ldots$$

For each k, we may write the partial sum

$$S_k(z) = \sum_{n=o}^{k} a_n z^n$$

$$= \sum_{n=o}^{k} z^n \frac{1}{2\pi i} \int_\gamma \frac{F(w)\,dw}{w^{n+1}}$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{F(w)}{w} \sum_{n=o}^{k} \left(\frac{z}{w}\right)^n dw$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{F(w)}{(w-z)} - \frac{1}{2\pi i} \int_\gamma \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw$$

$$= F(z) - \frac{1}{2\pi i} \int_\gamma \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw$$

The L(r,t)-transform of $S_k(z)$ is given by

$$(1-r)^{n+1} \exp(tr/1-r) \sum_{k=n}^{\infty} L^n_{k-n}(t)\, r^{k-n} \left[F(z) - \frac{1}{2\pi i} \int_\gamma \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw\right]$$

$= A - B$, where

$$A = (1-r)^{n+1} \exp(tr/(1-r)) \sum_{k=n}^{\infty} L^n_{k-n}(t)\, r^{k-n} F(z)$$

$$= (1-r)^{n+1} \exp(tr/(1-r)) \sum_{k=o}^{\infty} L^n_k(t)\, r^k F(z)$$

$= F(z)$, and

$$B = \frac{(1-r)^{n+1} \exp(tr/(1-r))}{2\pi i} \sum_{k=n}^{\infty} L^n_{k-n}(t)\, r^{k-n} \int_\gamma \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw$$

$$= \frac{(1-r)^{n+1} \exp(tr/(1-r))}{2\pi i} \sum_{k=o}^{\infty} L^n_k(t)\, r^k \int_\gamma \left(\frac{z}{w}\right)^{n+1} \frac{F(w)}{(w-z)} \left(\frac{z}{w}\right)^k dw.$$

Because of (10), the order of summation and integration may be changed, hence

$$B = \frac{(1 - r)^{n+1} \exp(tr/(1 - r))}{2\pi i} \int_\gamma \left(\frac{z}{w}\right)^{n+1} \frac{F(w)}{(w-z)} \sum_{k=0}^{\infty} L_k^n(t) \left(\frac{rz}{w}\right)^k dw$$

$$= \frac{(1 - r)^{n+1} \exp(tr/(1 - r))}{2\pi i} \int_\gamma \left(\frac{z}{w}\right)^{n+1} \frac{F(w)}{(w-z)} \left(1 - \frac{rz}{w}\right)^{-n-1} \exp\left(\frac{trz/w}{1 - rz/w}\right) dw$$

$$= \frac{\exp(tr/(1 - r))}{2\pi i} \int_\gamma \frac{F(w)}{(w-z)} \left(\frac{z - rz}{w - rz}\right)^{n+1} \exp\left(\frac{-trz}{w-rz}\right) dw \, .$$

With the help of (8), (9), (11), and the fact that γ has finite length, it becomes evident that $\lim_{n\to\infty} B = 0$; and hence $\lim_{n\to\infty} t_n = F(z)$, $z \in C(r)$.

4. We define the product of two methods of summability A_1, A_2 as the A_1-transform of the A_2-transform of the sequence $[S_k]$ and refer to it as the $A_1 \cdot A_2$ method of summability. If A_1, A_2 are regular methods of summability, then $A_1 \cdot A_2$ is regular. ([2], p. 83).

BOREL METHOD. Consider the sequence $[S_k]$. The B-transform of $[S_k]$ is the function

$$F(x) = e^{-x} \sum_{k=0}^{\infty} S_k \frac{x^k}{k!} \, .$$

If the power series $\sum_{k=0}^{\infty} S_k \frac{x^k}{k!}$ converges everywhere and if, for real values of x, $\lim_{x\to\infty} F(x) = S$, then the sequence $[S_k]$ is B-summable to the value S. The Borel Method is regular.

Theorem 4. If $0 < r < 1$, the series $\sum_{n=0}^{\infty} z^n$ is summable to $(1 - z)^{-1}$ by the iterated product method B. L(r,t) for all z in the region

$$(12) \qquad |z| < r^{-1} \cap \left|z - \left(\frac{r^{-1} + 1}{2}\right)\right| > \left(\frac{r^{-1} - 1}{2}\right)$$

Proof: Consider the geometric series $\sum_{n=0}^{\infty} z^n$ and the sequence of partial sums $[S_k]$ with $S_k(z) = \frac{1 - z^{k+1}}{1 - z}$, $z \neq 1$. The L(r,t)-transform of $S_k(z)$, $0 < r < 1$, is the sequence

$$t_n(z) = (1 - r)^{n+1} \exp\,(tr/(1-r)) \sum_{k=n}^{\infty} L_{k-n}^{n}(t)\, r^{k-n}\, S_k(z)$$

and it is clear that L(r,t) is applicable to $S_k(z)$ if and only if $|z| < r^{-1}$.

The B-transform of $[t_n(z)]$ is the function

$$F(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} (1 - r)^{n+1} \exp\,(tr/(1-r)) \sum_{k=n}^{\infty} L_{k-n}^{n}(t)\, r^{k-n} \frac{1-z^{k+1}}{1 - z}$$

$$= e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} (1 - r)^{n+1} \exp\,(tr/(1-r)) \sum_{k=0}^{\infty} L_{k}^{n}(t)\, r^{k} \frac{1-z^{k+n+1}}{1 - z}$$

$$= \frac{1}{1-z} - e^{-x} \exp\,(tr/(1-r)) \exp\,(-trz/(1-rz)) \frac{(1-r)z}{(1-z)(1-rz)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x(1-r)z}{1-rz}\right)^n$$

$$= \frac{1}{1-z} - e^{-x} \exp\,(tr/(1-r)) \exp\,(-trz/(1-rz)) \exp\,(x(1-r)z/(1-rz)) \frac{(1-r)z}{(1-z)(1-rz)}$$

$$(13)\quad = \frac{1}{1-z} - \exp\,(tr/(1-r)) \exp\,(-trz/(1-rz)) \exp\,(x(z-1)/(1-rz)) \frac{(1-r)z}{(1-z)(1-rz)}\,.$$

From (13) we see that in order for $\lim_{x\to\infty} F(x) = \frac{1}{1-z}$, z must be such that Re $((z-1)/(1-rz)) < 0$.

Put $z = u + lv$.

Then Re $((z-1)/(1-rz)) = \dfrac{(u - 1) - r(u^2 + v^2) + ru}{(1 - ru)^2 + r^2v^2}$.

This is negative provided, $(u - 1) - r(u^2 + v^2) + ru < 0$,

which is equivalent to $(u - (1 + r)/2r)^2 + v^2 > ((1 - r)/2r)^2$;

which in terms of z is given by $|z - (1+r)/2r| > ((1-r)/2r)$.

Therefore, the geometric series is summable to $(1 - z)^{-1}$ by the product B·L(r,t)-method for all z in the region (12).

Now, as before consider the power series $\sum_{n=o}^{\infty} a_n z^n$ with finite positive radius of convergence R, and let F(z) denote the function obtained by analytic continuation along radial lines from the origin.

Corresponding to each singularity ξ of F(z) and with r fixed, let $C(r,\xi)$ denote the set of points z for which

$$(14) \quad |z| < r^{-1}|\xi| \cap \left|\frac{z}{\lambda} - \frac{(r^{-1}\xi + \xi)}{2}\right| > \left|\frac{r^{-1}\xi - \xi}{2}\right|$$

whenever $\lambda \geqslant 1$, and let C(r) denote the intersection of the set $C(r,\xi)$.

Theorem 5. For each fixed r satisfying $0 < r < 1$, the series $\sum_{n=o}^{\infty} a_n z^n$ is summable B·L(r,t) to F(z) at every interior point z of the region C(r).

Proof: If z = 0, the theorem is trivially true. Suppose, then that $z \neq 0$ and z is a fixed point of C(r). Then $|rz| < |\xi|$ for all singularities ξ of F(z), and it follows that the circle C_1 with center at the origin and radius $|rz|$ is contained in the Mittag–Leffler star S of F. Also, when $\lambda \geqslant 1$,

$$\left|\frac{z}{\lambda\xi} - \frac{1 + r}{2r}\right| > \left|\frac{1 - r}{2r}\right|; \qquad \text{and, as we see by}$$

making inversion in which complex numbers are replaced by their reciprocals,

$$|\lambda\xi - \frac{z + rz}{2r}| > |\frac{z - rz}{2r}|.$$

This means that, whenever ξ is a singular point of F and $\lambda \geqslant 1$, the point $\lambda\xi$ lies outside the circle C_2 of points U for which

$$|U - \frac{z + rz}{2}| = |\frac{z - rz}{2}| \qquad \text{lies inside S.}$$

This implies that we can choose a simple closed curve γ which has finite length, which lies in S and which contains both C_1, C_2 in its interior.

Since the origin is in C_1 and z is on C_2, both of these points are inside γ. Inside and on γ F(z) is analytic. Consequently, for each w on γ the following inequalities hold:

(15) $|F(w)| \leqslant M, \qquad M = \max |F(w)| \qquad$ for w on γ;

(16) $|w - z| \geqslant N > 0, \qquad N = \min |w-z| \qquad$ for w on γ;

(17) $|rz| < |w|$;

(18) $|w - rz| \geqslant P > 0, \qquad P = \min |w - rz| \qquad$ for w on γ.

Returning now to the series $\sum_{n=o}^{\infty} a_n z^n$, the coefficients a_n are given by the Cauchy formula:

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{F(w)\, dw}{w^{n+1}} \qquad n = 0,1,2,...$$

For each k, we may write the partial sum

$$S_k(z) = \sum_{n=o}^{k} a_n z^n$$

$$= \sum_{n=o}^{k} z^n \frac{1}{2\pi i} \int_\gamma \frac{F(w)\, dw}{w^{n+1}}$$

$$= F(z) - \frac{1}{2\pi i} \int_\gamma \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw .$$

The B·L(r,t)-transform of $S_k(z)$ is therefore, given by

$$= e^{-x} \sum_{n=o}^{\infty} \frac{x^n}{n!} (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) \sum_{k=n}^{\infty} L_{k-n}^{n}(t)\, r^{k-n} \left[F(z) - \frac{1}{2\pi i} \int_\gamma \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw\right]$$

(19) $= F(z) - R_z(x)$, where $R_z(x)$ is given by

$$= e^{-x} \sum_{n=o}^{\infty} \frac{x^n}{n!} (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) \sum_{k=n}^{\infty} L_{k-n}^{n}(t)\, r^{k-n} \frac{1}{2\pi i} \int_\gamma \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw$$

$$= e^{-x} \sum_{n=o}^{\infty} \frac{x^n}{n!} (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) \sum_{k=o}^{\infty} L_{k}^{n}(t)\, r^{k} \frac{1}{2\pi i} \int_\gamma \frac{F(w)}{(w-z)} \left(\frac{z}{w}\right)^{n+1} \left(\frac{z}{w}\right)^k dw$$

Because of (17), the order of summation and integration may be changed, hence

$$R_z(x) = \frac{e^{-x}}{2\pi i} \sum_{n=0}^{\infty} \frac{x^n}{n!} (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) \int_\gamma \left(\frac{z}{w}\right)^{n+1} \frac{F(w)}{(w-z)} \sum_{k=0}^{\infty} L_k^n\ (t) \left(\frac{rz}{w}\right)^k dw.$$

From (19), if is apparant that $F(x) \to F(z)$ provided $R_z(x) \to 0$ as $x \to \infty$. With the help of (15), (16), (18) and the fact that γ has finite length it becomes evident that $\lim_{x\to\infty} R_z(x) = 0$, provided

$$\mathrm{Re}\left(\frac{z - w}{w - zr}\right) < 0, \text{ which is equivalent to}$$

$$\left|w - \frac{1 + r}{2} z\right| > \frac{1 - r}{2} |z|.$$

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